UNIVERSITÉ DE TECHNOLOGIE TROYES

Benchmark of (some) AP schemes for the hyperbolic to diffusive limit

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with contributors

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Outline

1. Introduction

2. Presentation of the schemes used
   - HLL-SRC
   - HLL-AP
   - HLL-θ
   - AHO/TAHO

3. Results
   - Riemann problem
   - Continuous solution
   - Convergence to the diffusion
     - Increasing σt
     - Late-time behaviour

4. Conclusion and perspectives

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Context

System of conservation laws with source term:

$$\partial_t \mathbf{W} + \operatorname{div}(\mathbf{F}(\mathbf{W})) = \gamma(\mathbf{W})(\mathbf{R}(\mathbf{W}) - \mathbf{W}) \tag{1}$$

- $\mathcal{A}$: set of admissible states,
- $\mathbf{W} \in \mathcal{A} \subset \mathbb{R}^N$: conservative variables,
- $\mathbf{F}$: physical flux,
- $\gamma > 0$: parameter of the stiffness,
- $\mathbf{R}$: continuous function which satisfies the conditions from Berthon, LeFloch, and Turpault [BLT13].

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Under the compatibility conditions, (1) degenerates to a diffusion equation when $\gamma t \to \infty$:

$$\partial_t w - \mathrm{div}\big(f(w)\nabla w\big) = 0. \tag{2}$$

- $w \in \mathbb{R}$ linked to $\mathbf{W}$ and $f(w) > 0$.

Telegraph equations

$$\begin{cases} \partial_t u + \partial_x a u = \sigma(v - u) \\ \partial_t v - \partial_x a v = \sigma(u - v) \end{cases}$$

$$\mathcal{A} = \{(u, v)^T \in \mathbb{R}^2\}$$

Formalism of (1):

- $\mathbf{W} = (u \quad v)^T$
- $\mathbf{R}(\mathbf{W}) = (v \quad u)^T$
- $\mathbf{F}(\mathbf{W}) = a(u \quad -v)^T$
- $\gamma(\mathbf{W}) = \sigma > 0$

Example #1

Telegraph equations

$$\begin{cases} \partial_t u + \partial_x a u = \sigma(v - u) \\ \partial_t v - \partial_x a v = \sigma(u - v) \end{cases}$$

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Limit ($\sigma t \to \infty$):

$$\partial_t(u+v) - \partial_x\left(\frac{a^2}{2\sigma}\partial_x(u+v)\right) = 0$$

Example #1'

Telegraph equations

$$\begin{cases} \partial_t z + \partial_x a w = 0 \\ \partial_t w + \partial_x a z = -2\sigma w \end{cases}$$

$$\mathcal{A} = \{(z, w)^T \in \mathbb{R}^2\}$$

Formalism of (1):

- $\mathbf{W} = (z \quad w)^T$
- $\mathbf{F}(\mathbf{W}) = a(w \quad z)^T$
- $\mathbf{R}(\mathbf{W}) = (z \quad 0)^T$
- $\gamma(\mathbf{W}) = 2\sigma > 0$

Limit ($\sigma t \to \infty$):

$$\partial_t(z) - \partial_x\left(\frac{a^2}{2\sigma}\partial_x(z)\right) = 0$$

Example #1”

Telegraph equations

$$\begin{cases} \partial_t z + \varepsilon^{-1}\partial_x a w = 0 \\ \partial_t w + \varepsilon^{-1}\partial_x a z = -2\varepsilon^{-2}\sigma w \end{cases}$$

$$\mathcal{A} = \{(z, w)^T \in \mathbb{R}^2\}$$

Formalism of (1):

- $\mathbf{W} = (z \quad w)^T$
- $\mathbf{R}(\mathbf{W}) = (z \quad 0)^T$
- $\mathbf{F}(\mathbf{W}) = a\varepsilon^{-1}(w \quad z)^T$
- $\gamma(\mathbf{W}) = 2\varepsilon^{-2}\sigma > 0$

Limit ($\varepsilon \to 0$):

$$\partial_t(z) - \partial_x\left(\frac{a^2}{2\sigma}\partial_x(z)\right) = 0$$

P_1 model for radiative transfer:

$$\begin{cases} \partial_t E_R + \operatorname{div}(\mathbf{F}_R) = 0 \\ \partial_t \mathbf{F}_R + \dfrac{c^2}{3} \nabla E_R = -c\sigma^d \mathbf{F}_R \end{cases}$$

$$\mathcal{A} = \{(E_R, \mathbf{F}_R)^T \in \mathbb{R}^3\}$$

Formalism of (1):

- $\mathbf{W} = (E_R, \mathbf{F}_R)^T$
- $\mathbf{R}(\mathbf{W}) = (E_R, 0)^T$
- $\mathbf{F}(\mathbf{W}) = \left(\mathbf{F}_R, \frac{c^2}{3} E_R \mathbf{I}\right)^T$
- $\gamma(\mathbf{W}) = c\sigma^d$

Example #2

P_1 model for radiative transfer:

$$\begin{cases} \partial_t E_R + \mathrm{div}(\mathbf{F}_R) = 0 \\ \partial_t \mathbf{F}_R + \dfrac{c^2}{3} \nabla E_R = -c\sigma^d \mathbf{F}_R \end{cases}$$

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- $\gamma(\mathbf{W}) = c\sigma^d$

Limit ($c\sigma^d t \to \infty$):

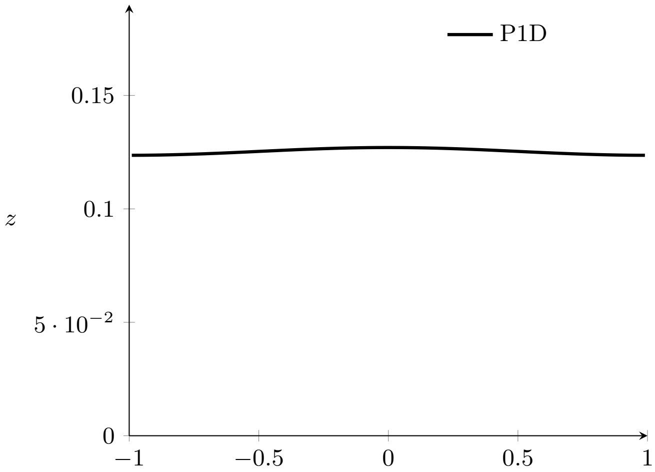
$$\partial_t E_R - \mathrm{div}\left(\frac{c}{3\sigma^d} \nabla E_R\right) = 0$$

Advantages of an AP scheme #1

Gaussian:

$$z_0(x) = u_0(x) + v_0(x) = \exp(-50x^2) \; ; w_0(x) = u_0(x) - v_0(x) = 0$$

$$\sigma = 10^0 \; ; t_f = 1 \; ; \Delta x = 2 \times 10^{-2}$$



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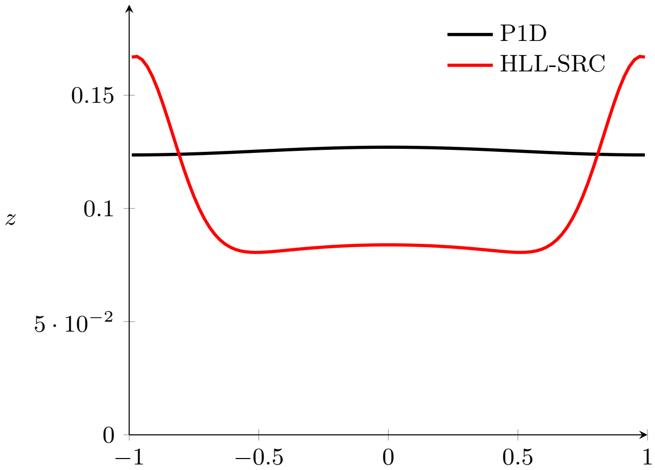

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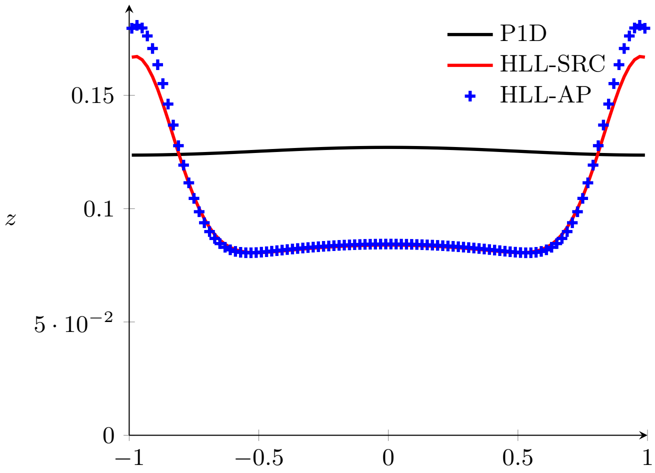

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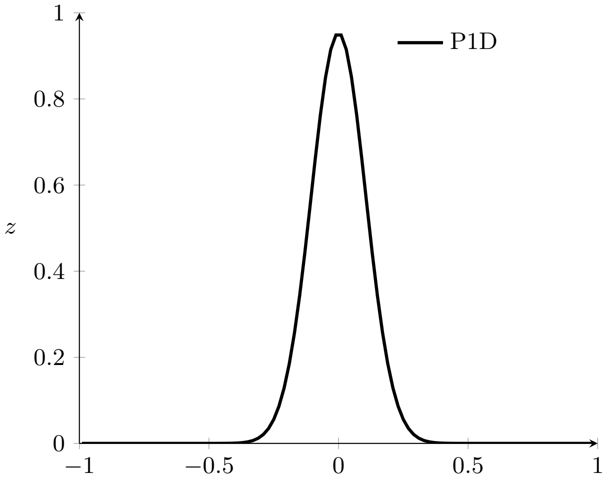

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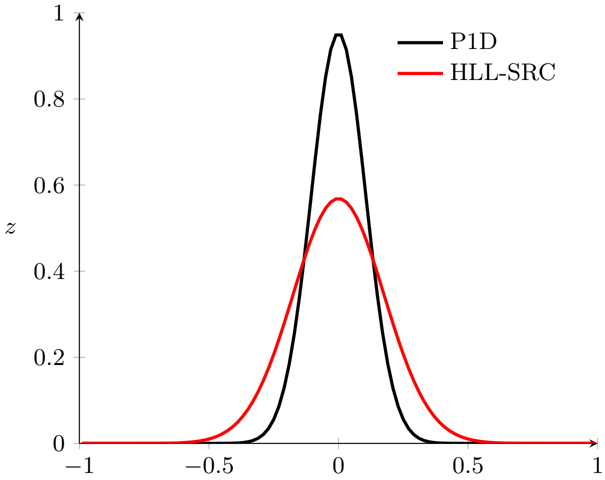

Advantages of an AP scheme #2

Gaussian:

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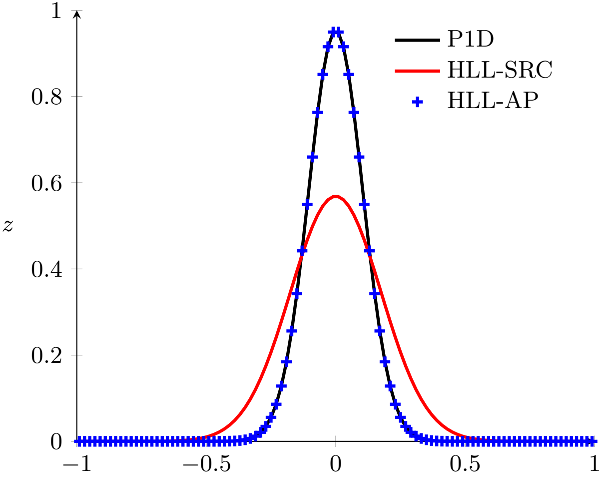

Advantages of an AP scheme #3

Gaussian:

$$z_0(x) = u_0(x) + v_0(x) = \exp(-50x^2) \; ; w_0(x) = u_0(x) - v_0(x) = 0$$
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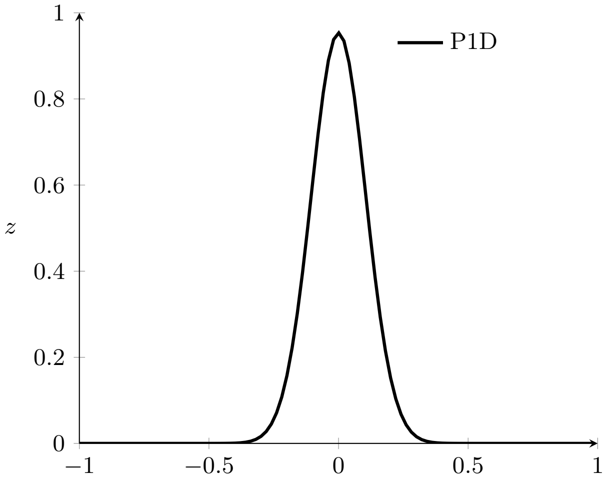

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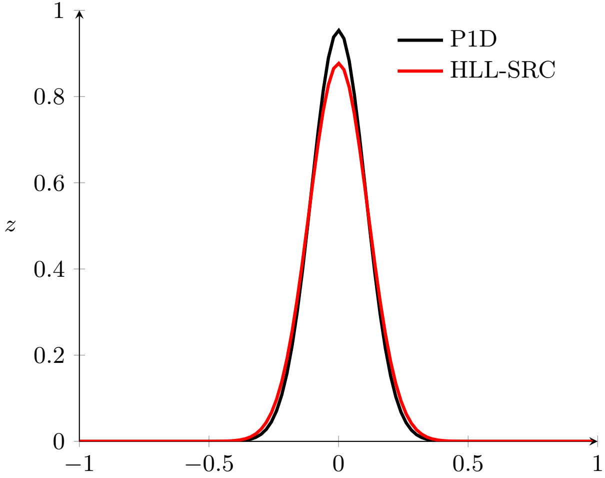

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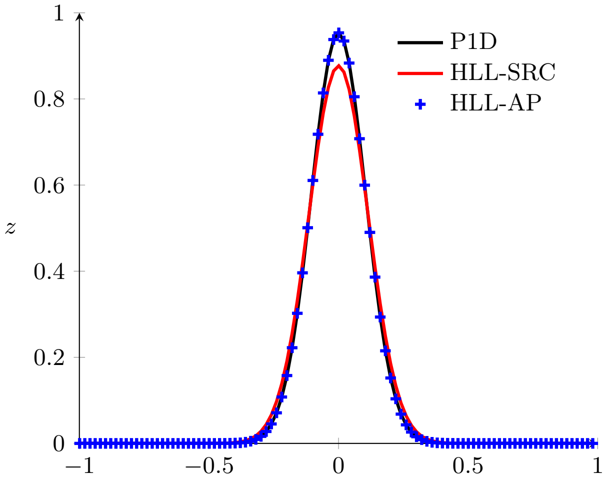

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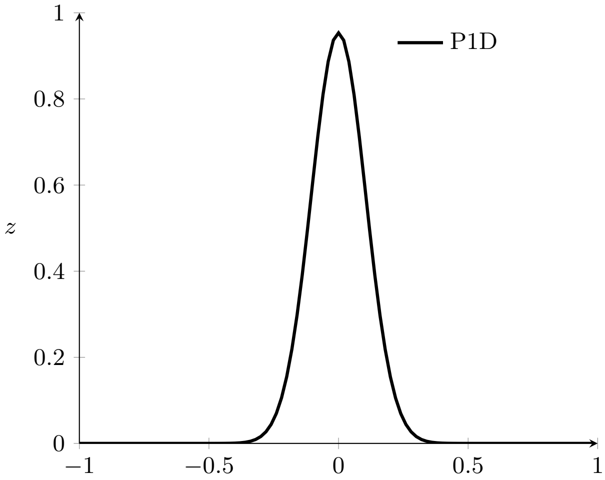

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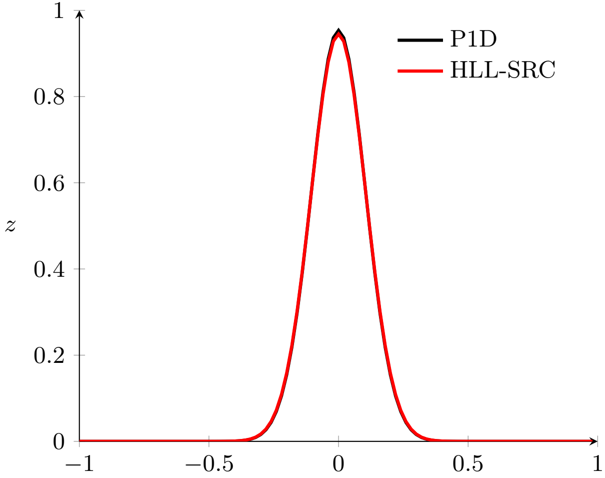

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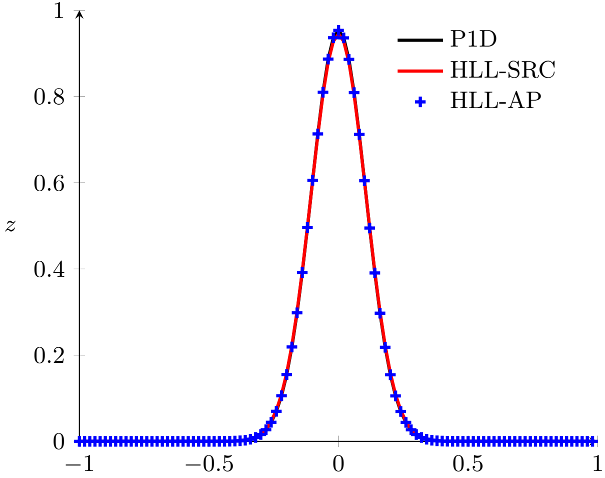

Conservation laws (1) :
$$\partial_t \mathbf{W} + \mathrm{div}(\mathbf{F}(\mathbf{W})) = \gamma(\mathbf{W})(\mathbf{R}(\mathbf{W}) - \mathbf{W})$$

$\gamma t \to \infty$

Diffusion equation (2) :
$$\partial_t w - \mathrm{div}\big(f(w)\nabla w\big) = 0$$

consistant :
$\Delta t, \Delta x \to 0$

consistant ?

Numerical scheme

$\gamma t \to \infty$

Limit scheme

Possible schemes (non exhaustive list)

1. Rusanov/[HLL83] with explicit centered discretisation of the source term,
2. Jin and Levermore [JL96],
3. full implicit or IMEX [BPR13],
4. Gosse and Toscani [GT02],
5. AHO/TAHO from Aregba-Driollet, Briani, and Natalini [ABN08],
6. generalisation of [GT02] by Berthon and Turpault [BT11].
7. inspired by low-Mach scheme from [Gir14; CGK16] by Chalons *et al.* in [CG17; CT18],
8. ...

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$$W_i^{n+1} = W_i^n - \frac{\Delta t}{\Delta x}(\mathcal{F}_{i+1/2}^n - \mathcal{F}_{i-1/2}^n) + \Delta t S(W_i^n), \tag{3}$$

- $\mathcal{F}_{i+1/2}^n$: standard HLL
- CFL condition:

$$\Delta t \leq \frac{\Delta x}{4(a + 2\sigma \Delta x)}$$

Limit

$$
\begin{aligned}
0 &= z_{i+1}^{n,0} - 2z_i^{n,0} + z_{i+1}^{n,0} \\
w_i^{n,0} &= 0 \\
z_i^{n+1,0} &= z_i^{n,0} + \frac{\Delta t}{\Delta x^2}\frac{a^2}{2\sigma}\left(\frac{z_{i+2}^{n,0} - 2z_i^{n,0} + z_{i-2}^{n,0}}{4}\right) + \frac{a^2\Delta t}{2\sigma\Delta x}\frac{\sigma\Delta x}{a}\left(z_{i+1}^{n,1} - 2z_i^{n,1} + z_{i+1}^{n,1}\right)
\end{aligned}
$$

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$$W_i^{n+1} = W_i^n - \frac{\Delta t}{\Delta x}(\alpha_{i+1/2}\mathcal{F}_{i+1/2} - \alpha_{i-1/2}\mathcal{F}_{i-1/2})$$
$$+\frac{\Delta t}{\Delta x}\left((1-\alpha_{i-1/2})S_{i-1/2} + (1-\alpha_{i+1/2})S_{i+1/2}\right), \quad (4)$$

with

$$\alpha_{i+1/2} = \frac{2b_{i+1/2}}{2b_{i+1/2} + (\sigma + \bar{\sigma})\Delta x} = 1 + \mathcal{O}(\Delta x) \in [0;1],$$

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{\Delta x}(\alpha_{i+1/2}\mathcal{F}_{i+1/2} - \alpha_{i-1/2}\mathcal{F}_{i-1/2}) + \frac{\Delta t}{\Delta x}\left((1-\alpha_{i-1/2})S_{i-1/2} + (1-\alpha_{i+1/2})S_{i+1/2}\right), \tag{4}$$

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$$\alpha_{i+1/2} = \frac{2b_{i+1/2}}{2b_{i+1/2} + (\sigma + \bar{\sigma})\Delta x} = 1 + \mathcal{O}(\Delta x) \in [0;1],$$

- $\mathcal{F}^n_{i+1/2}$: standard HLL
- classical hyperbolic CFL condition:

$$\Delta t \leq \frac{1}{2}\frac{\Delta x}{a}$$

Limit

$$w_i^{n,0} = 0$$

$$z_i^{n+1,0} = z_i^{n,0} + \frac{\Delta t}{\Delta x^2} \frac{a^2}{2\sigma} \left(z_{i+1}^{n,0} - 2z_i^{n,0} + z_{i-1}^{n,0} \right)$$

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- numerical flux:

$$\mathcal{F}_{i+1/2}^n = \frac{1}{2}\Big(F(W_{i+1}) + F(W_i) - \theta_{i+1/2}(W_{i+1} - W_i)\Big), \tag{7}$$

with

$$\theta_{i+1/2} = \frac{1}{1+\sigma t} = \mathcal{O}(\frac{1}{\sigma t})$$

HLL-θ [CG17; CT18]

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{\Delta x}(\mathcal{F}_{i+1/2} - \mathcal{F}_{i-1/2}) + \Delta t S(W_i), \tag{5}$$

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  with

$$\theta_{i+1/2} = \frac{1}{1+\sigma t} = \mathcal{O}(\frac{1}{\sigma t})$$

- CFL condition:

$$\Delta t \leq \frac{\Delta x}{4(a + 2\sigma \Delta x)}$$

  or:

$$\Delta t \leq \min\left(\frac{\Delta x}{2a\theta}, \frac{2\sigma \Delta x^2}{a^2}\right)$$

HLL-θ, AP ?

Limit

$$w_i^{n,0} = 0$$

$$z_i^{n+1,0} = z_i^{n,0} + \frac{\Delta t}{\Delta x^2}\frac{a^2}{2\sigma}\left(\frac{z_{i+2}^{n,0} - 2z_i^{n,0} + z_{i-2}^{n,0}}{4}\right)$$

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$$W_i^{n+1} = W_i^n - \frac{\Lambda \Delta t}{2\Delta x}(W_{i+1}^n - W_{i-1}^n) + \frac{Q\Delta t}{2\Delta x}(W_{i+1}^n - 2W_i^n + W_{i-1}^n) \\ + \Delta t(B_{-1}W_{i-1}^n + B_0 W_i^n + B_{+1}W_{i+1}^n), \tag{8}$$

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- Λ, Q for [Rus61]: $\Lambda = a \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $Q = a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,
- consistency of the source term: $B_{-1} + B_0 + B_{+1} = S(W) + \Gamma + \Delta x C$,

$$W_i^{n+1} = W_i^n - \frac{\Lambda \Delta t}{2\Delta x}(W_{i+1}^n - W_{i-1}^n) + \frac{Q\Delta t}{2\Delta x}(W_{i+1}^n - 2W_i^n + W_{i-1}^n) + \Delta t(B_{-1}W_{i-1}^n + B_0 W_i^n + B_{+1}W_{i+1}^n), \tag{8}$$

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- consistency of the source term: $B_{-1} + B_0 + B_{+1} = S(W) + \Gamma + \Delta x C$,
- CFL condition:

$$\Delta x < \frac{2}{\sigma} \quad \text{and} \quad \Delta t \leq \frac{\Delta x}{a + \Delta x \sigma (a^2 + 1)}$$

Limit

$$w_i^{n,0} = 0$$

$$z_i^{n+1,0} = z_i^{n,0} + \frac{\Delta t}{\Delta x^2}\frac{a^2}{2\sigma}\left(z_{i+1}^{n,0} - 2z_i^{n,0} + z_{i-1}^{n,0}\right) + \mathcal{O}(\sigma\Delta x^2)$$

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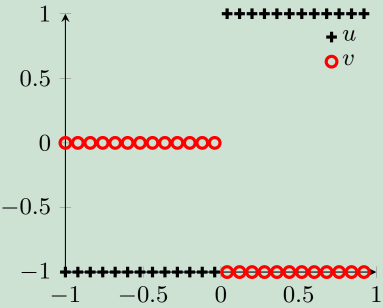
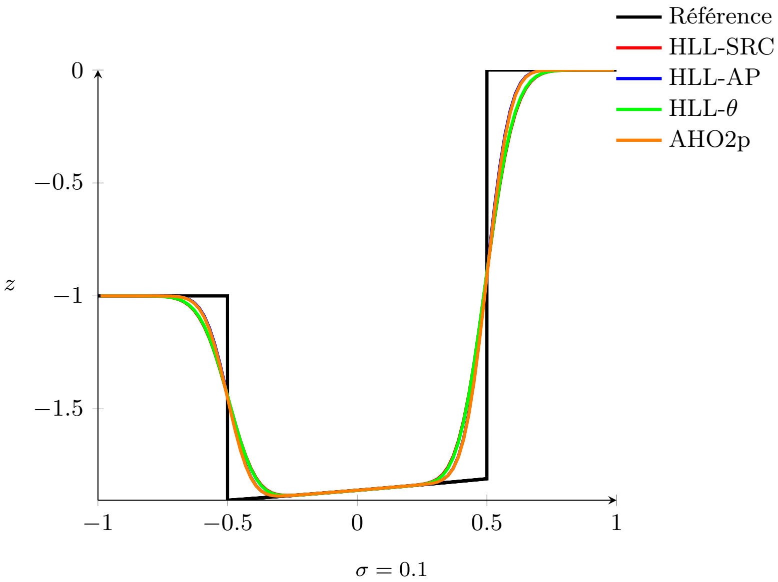
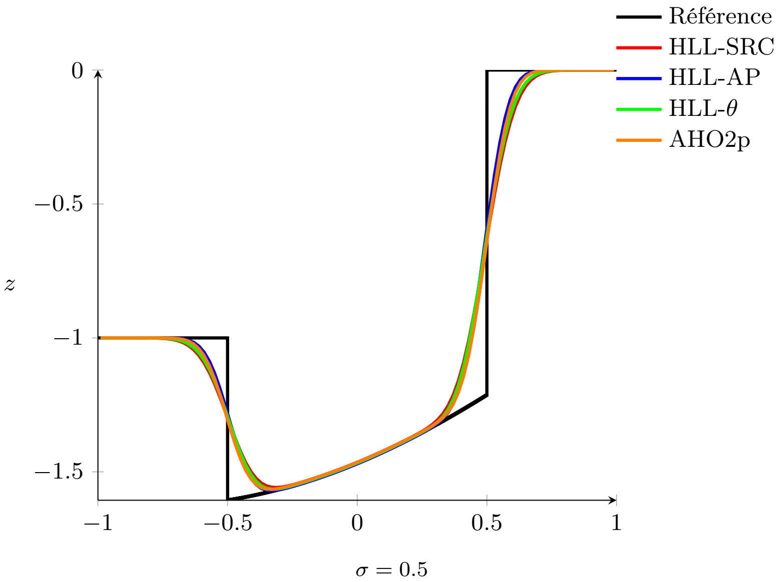
Computations of the solution in [Bla16]

$$u(0,x<0) = -1 \quad ; \quad u(0,x>0) = 1$$
$$v(0,x<0) = 0 \quad ; \quad v(0,x>0) = -1 \qquad (9)$$



- $x_{\max} = -x_{\min} = 1$,
- $a = 0.5$, $T_f = 1$,
- $\Delta x = 2 \times 10^{-2}$,
- Neumann boundary conditions.


Référence
HLL-SRC
HLL-AP
HLL-θ
AHO2p
0
−0.5
−1
−1.5
z
−1
−0.5
0
0.5
1
σ = 0.1


Référence
HLL-SRC
HLL-AP
HLL-θ
AHO2p
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−1
−1.5
z
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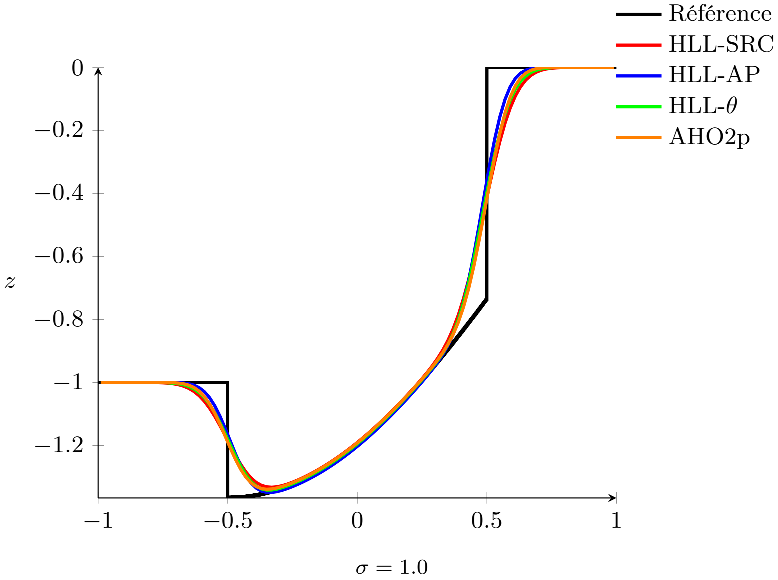
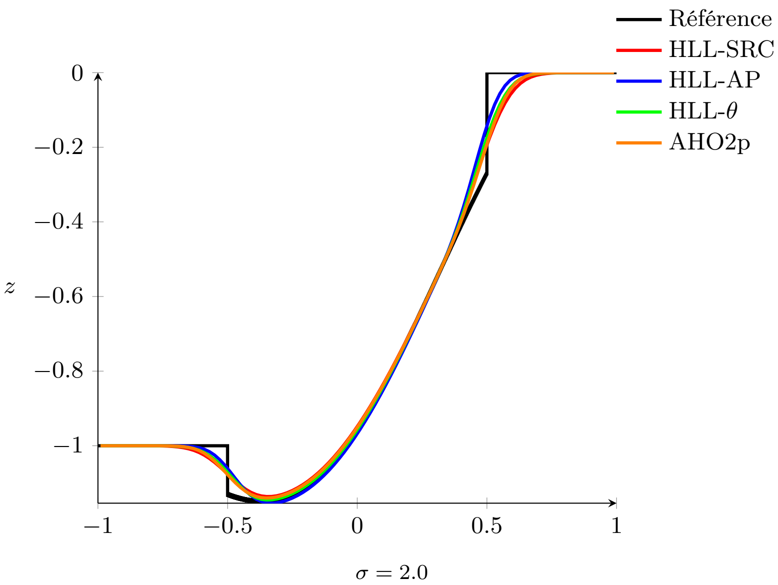
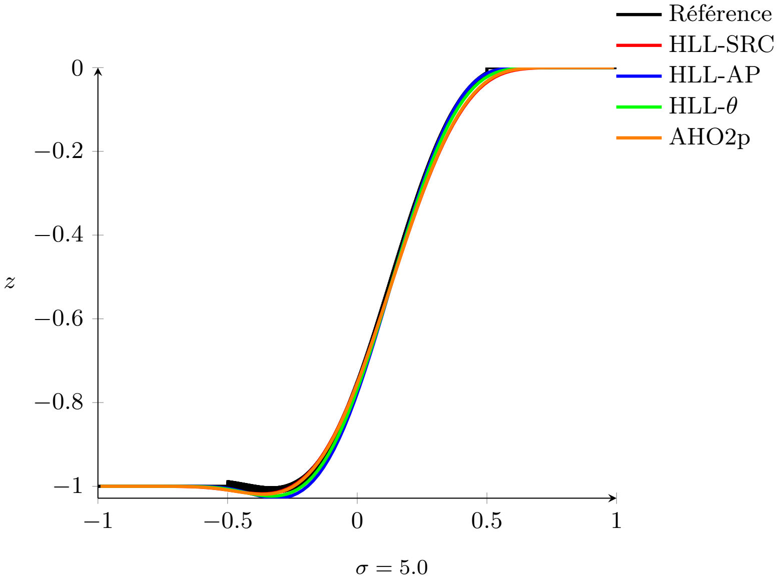
−0.5
0
0.5
1
σ = 0.5


Référence
HLL-SRC
HLL-AP
HLL-θ
AHO2p
0
−0.2
−0.4
−0.6
−0.8
−1
−1.2
z
−1
−0.5
0
0.5
1
σ = 1.0


Référence
HLL-SRC
HLL-AP
HLL-θ
AHO2p
0
−0.2
−0.4
−0.6
−0.8
−1
z
−1
−0.5
0
0.5
1
σ = 2.0


Référence
HLL-SRC
HLL-AP
HLL-θ
AHO2p
0
−0.2
−0.4
−0.6
−0.8
−1
z
−1
−0.5
0
0.5
1
σ = 5.0

1. Introduction

2. Presentation of the schemes used
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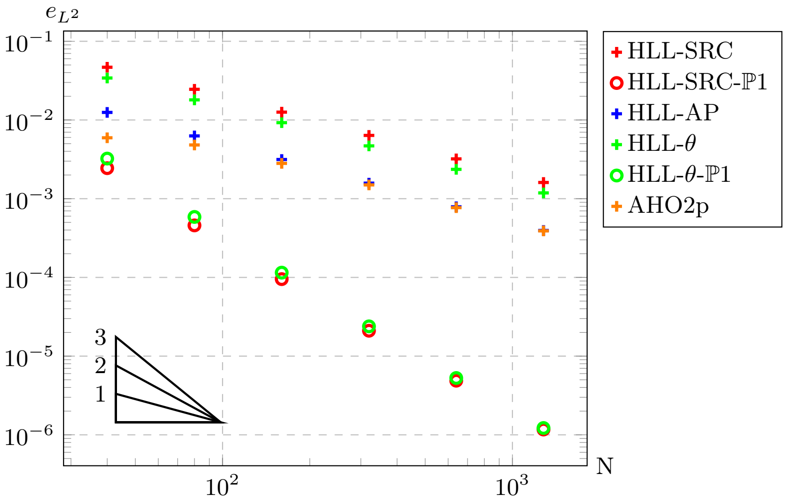
4. Conclusion and perspectives

From [BDF12; BT17; CT18]

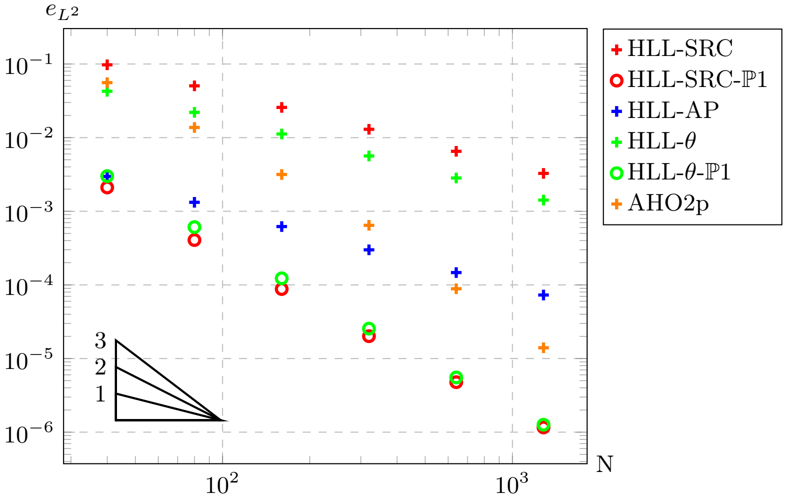
$$
\begin{aligned}
z(t,x) &= \frac{1}{\sigma}\cos\left(\frac{2\pi x}{L}\right)\Big[(\sigma+\omega)\exp(t(\omega-\sigma)) + (\sigma-\omega)\exp(-t(\omega+\sigma))\Big] \\
w(t,x) &= \frac{1}{\sigma}\frac{2\pi a}{L}\sin\left(\frac{2\pi x}{L}\right)\Big[\exp(t(\omega-\sigma)) + \exp(-t(\omega+\sigma))\Big]
\end{aligned}
\tag{10}
$$

- $L = x_{\max} - x_{\min}$, $\omega = \sigma^2 - \left(\frac{2\pi a}{L}\right)^2 > 0$,
- $x_{\min} = 0$, $x_{\max} = 1$,
- $a = 1$,
- periodic boundary conditions.

Continuous solution

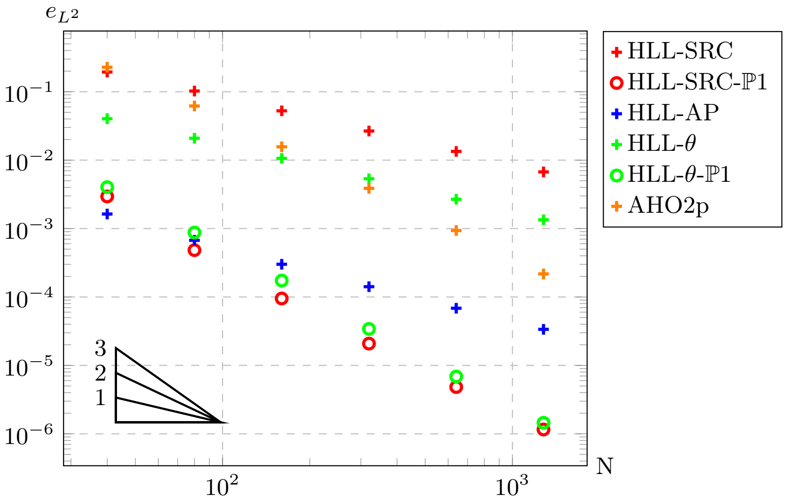


$\sigma = 8$, $T_f = 0.1$



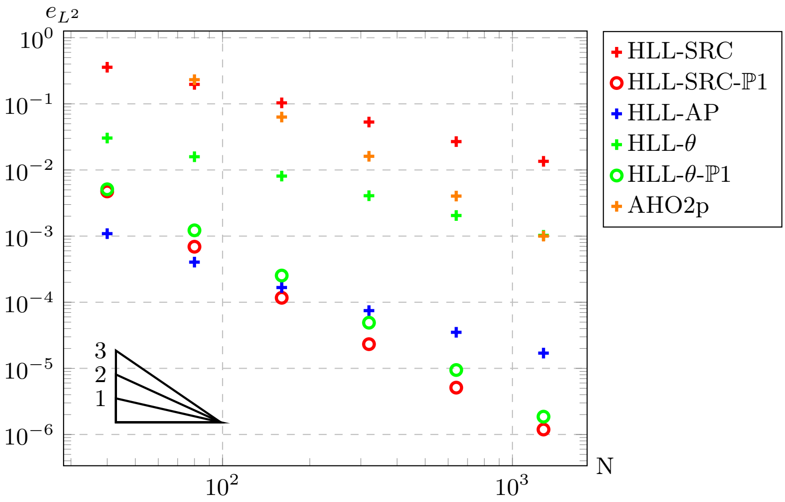
$\sigma = 16$, $T_f = 0.2$

Continuous solution

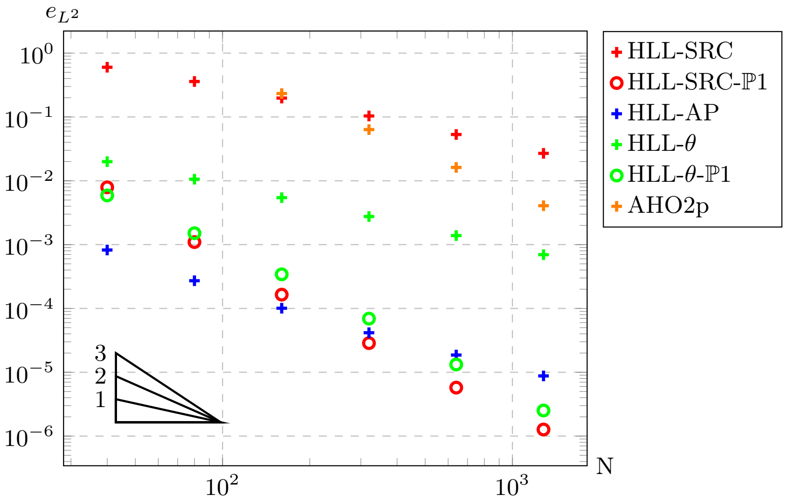


$\sigma = 32$, $T_f = 0.4$

Continuous solution

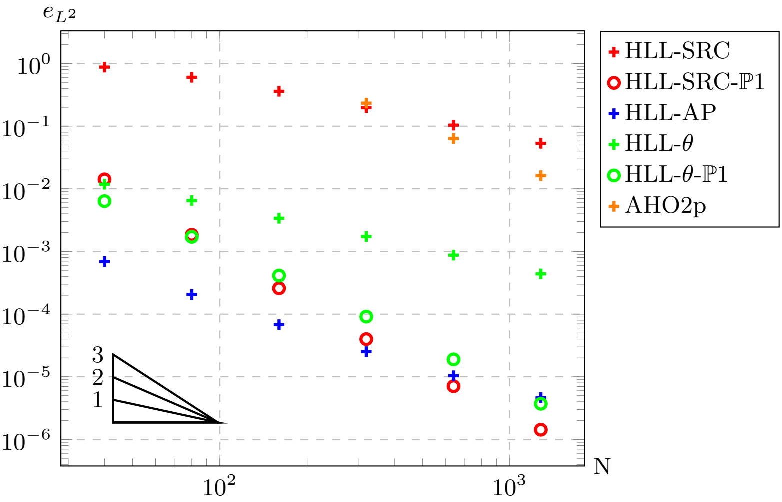


$\sigma = 64$, $T_f = 0.8$

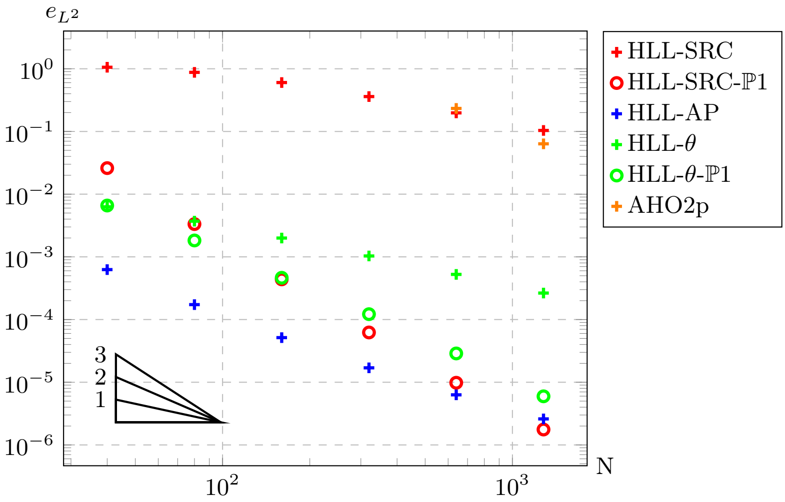


$\sigma = 128$, $T_f = 1.6$

Continuous solution

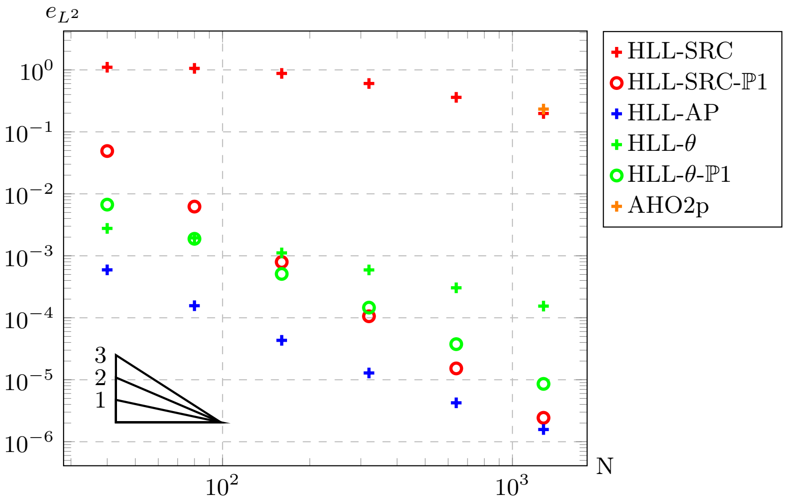


$\sigma = 256$, $T_f = 3.2$



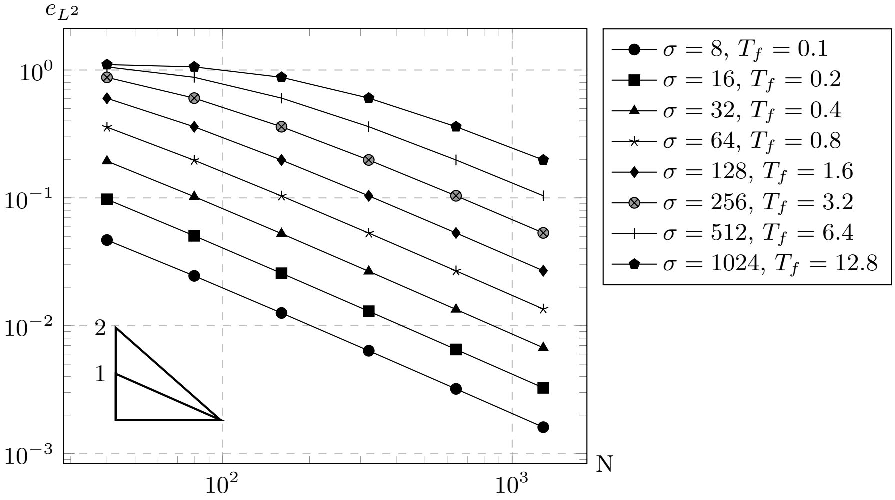
$\sigma = 512$, $T_f = 6.4$

Continuous solution



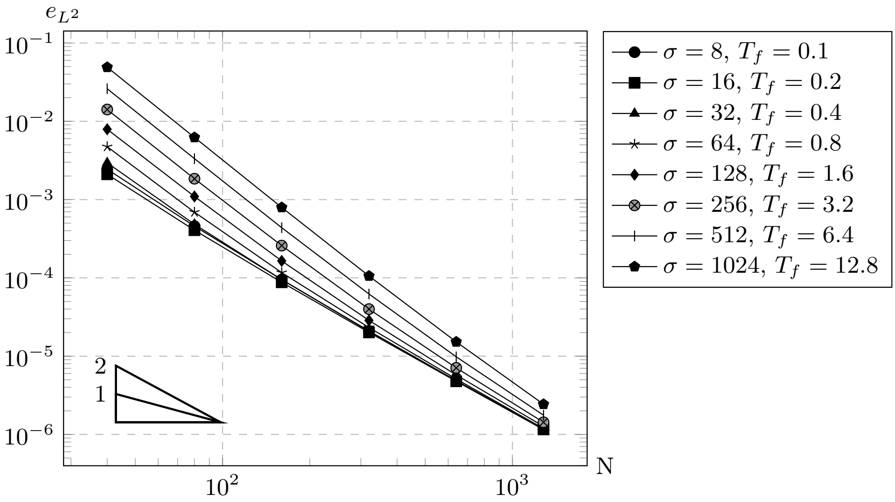
$\sigma = 1024$, $T_f = 12.8$

Continuous solution



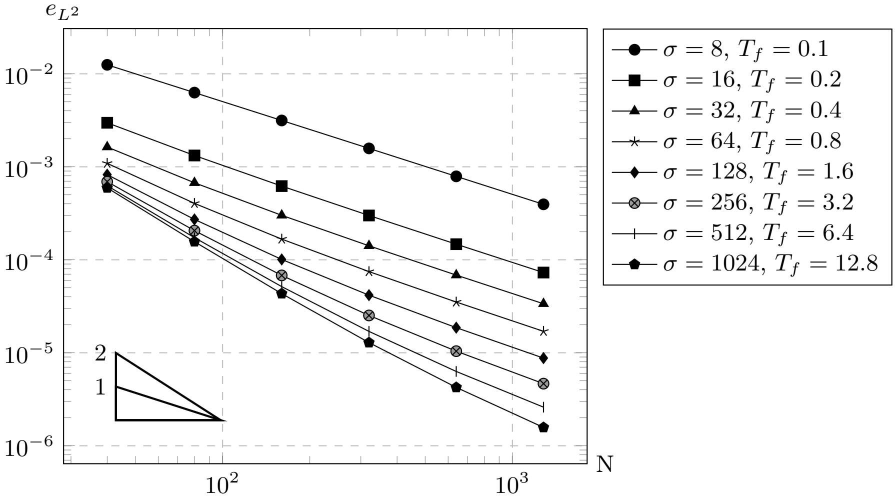
HLL-SRC

Continuous solution



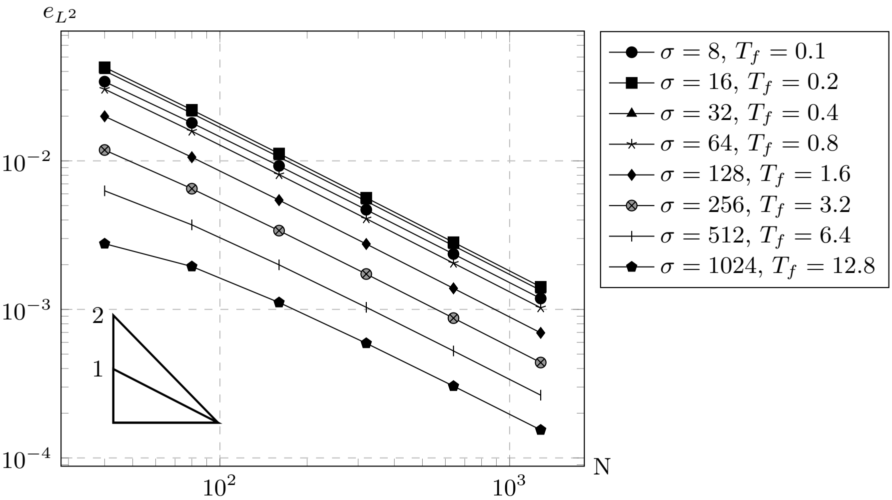
HLL-SRC-$\mathbb{P}1$

Continuous solution

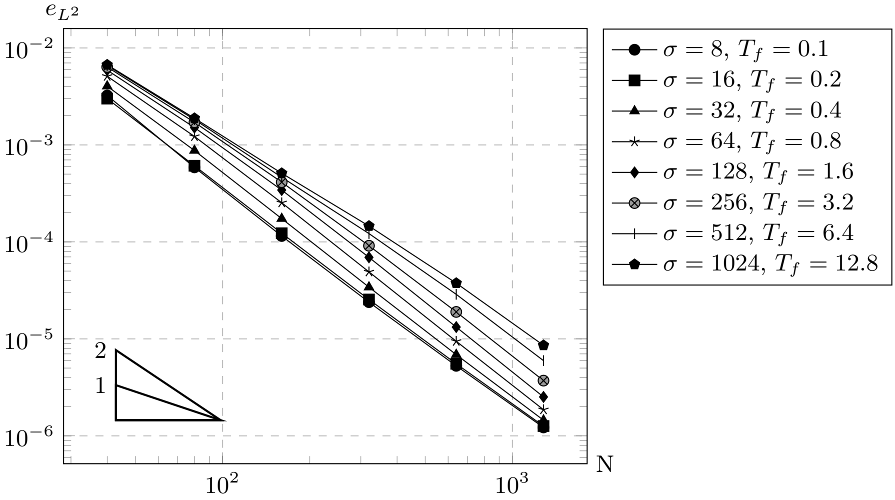


HLL-AP

Continuous solution



HLL-θ



HLL-θ-$\mathbb{P}1$

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4. Conclusion and perspectives

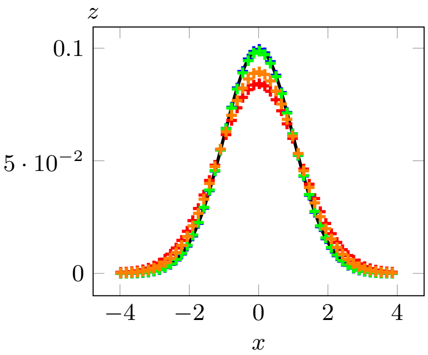
Convergence rates from [BHN07]

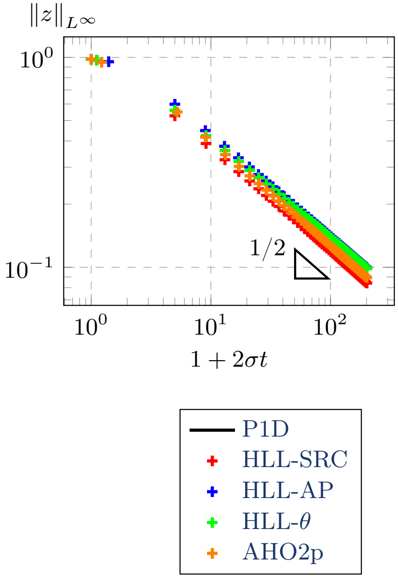
$$
\begin{aligned}
\|\partial_x^\beta \partial_t z\|_{L_p} &= \mathcal{O}(1+\sigma t)^{-\frac{n}{2}(1-1/p)-\beta/2-1/2} \\
\|\partial_x^\beta z\|_{L_p} &= \mathcal{O}(1+\sigma t)^{-\frac{n}{2}(1-1/p)-\beta/2} \\
\|\partial_x^\beta \partial_t w\|_{L_p} &= \mathcal{O}(1+\sigma t)^{-\frac{n}{2}(1-1/p)-\beta/2-1} \\
\|\partial_x^\beta w\|_{L_p} &= \mathcal{O}(1+\sigma t)^{-\frac{n}{2}(1-1/p)-\beta/2-1/2} \\
\|\partial_x^\beta (z-z_D)\|_{L_p} &= \mathcal{O}(1+\sigma t)^{-\frac{n}{2}(1-1/p)-\beta/2-1/2}
\end{aligned} \tag{11}
$$

Gaussian:

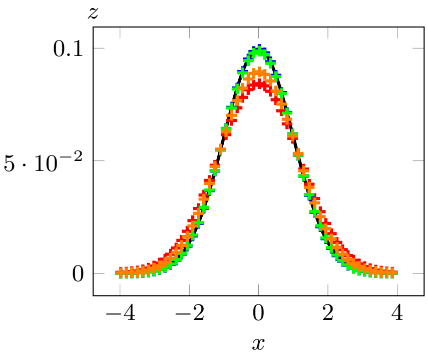
$$
\begin{aligned}
z(0,x) &= \exp(-50x^2) \\
w(0,x) &= 0
\end{aligned} \tag{12}
$$

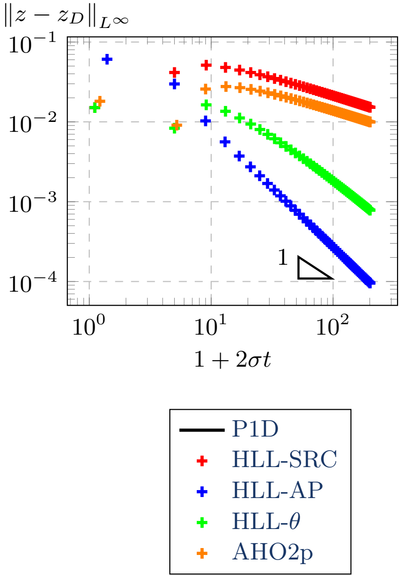
Increasing σt



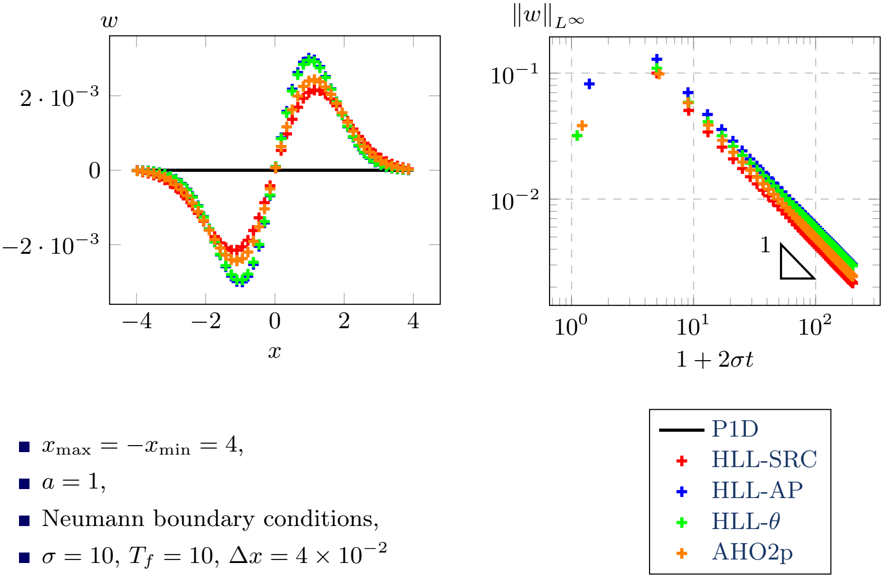


- $x_{\max} = -x_{\min} = 4$,
- $a = 1$,
- Neumann boundary conditions,
- $\sigma = 10$, $T_f = 10$, $\Delta x = 4 \times 10^{-2}$



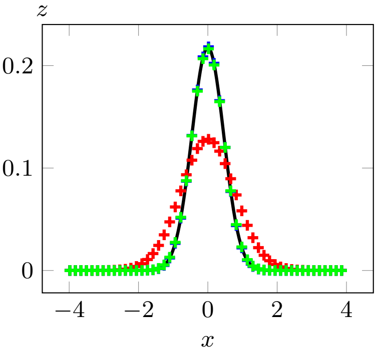


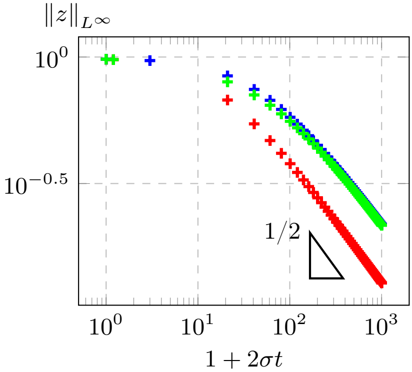
- $x_{\max} = -x_{\min} = 4$,
- $a = 1$,
- Neumann boundary conditions,
- $\sigma = 10$, $T_f = 10$, $\Delta x = 4 \times 10^{-2}$

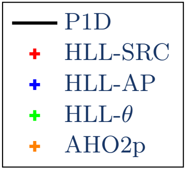


- $x_{\max} = -x_{\min} = 4$,
- $a = 1$,
- Neumann boundary conditions,
- $\sigma = 10$, $T_f = 10$, $\Delta x = 4 \times 10^{-2}$

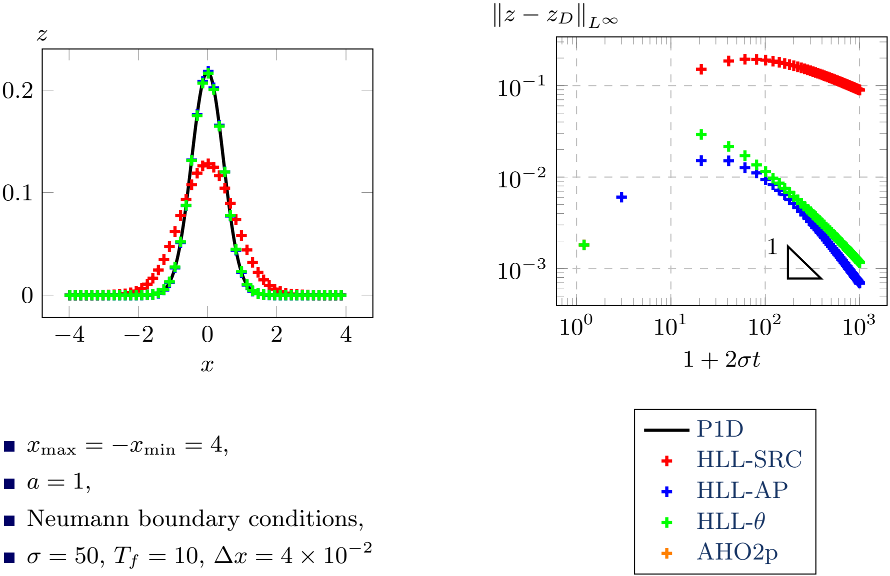
Increasing σt



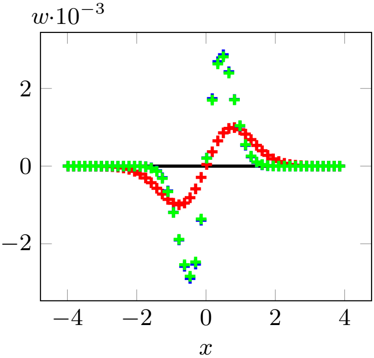


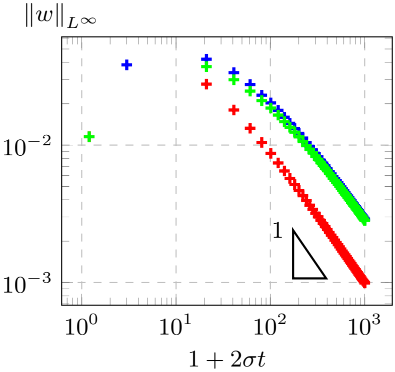


- $x_{\max} = -x_{\min} = 4$,
- $a = 1$,
- Neumann boundary conditions,
- $\sigma = 50$, $T_f = 10$, $\Delta x = 4 \times 10^{-2}$

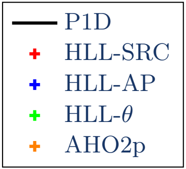


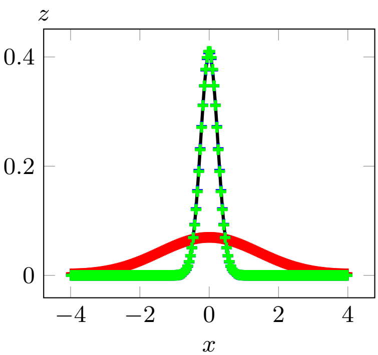
- $x_{\max} = -x_{\min} = 4$,
- $a = 1$,
- Neumann boundary conditions,
- $\sigma = 50$, $T_f = 10$, $\Delta x = 4 \times 10^{-2}$

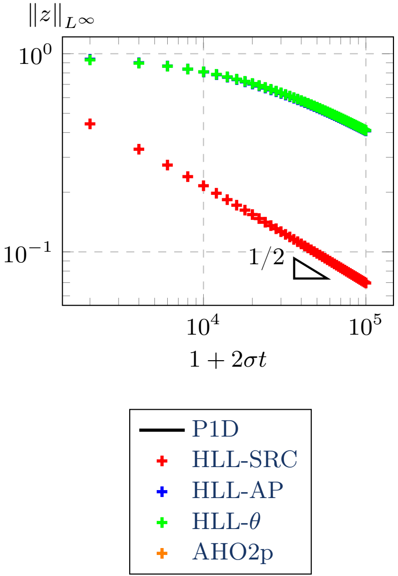




- $x_{\max} = -x_{\min} = 4$,
- $a = 1$,
- Neumann boundary conditions,
- $\sigma = 50$, $T_f = 10$, $\Delta x = 4 \times 10^{-2}$







- $x_{\max} = -x_{\min} = 4$,
- $a = 1$,
- Neumann boundary conditions,
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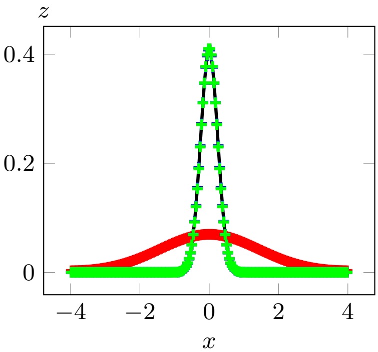
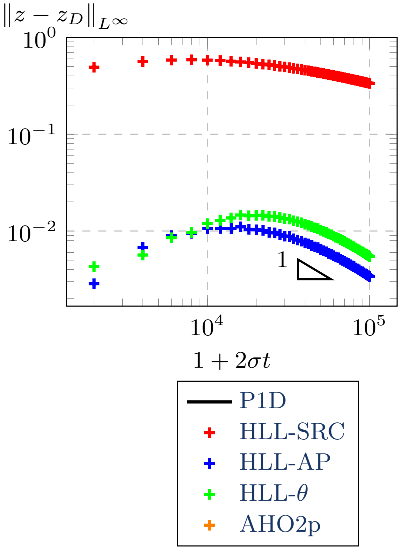
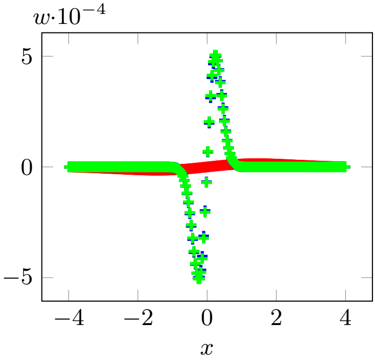
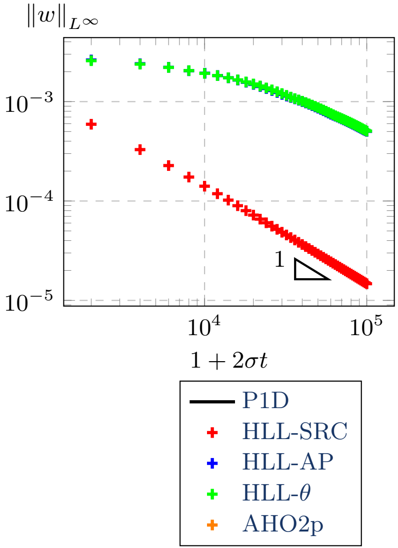




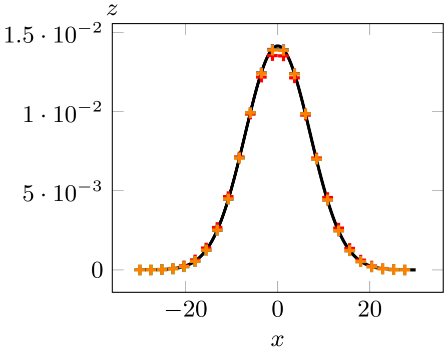
- $x_{\max} = -x_{\min} = 4$,
- $a = 1$,
- Neumann boundary conditions,
- $\sigma = 1000$, $T_f = 50$, $\Delta x = 4 \times 10^{-2}$

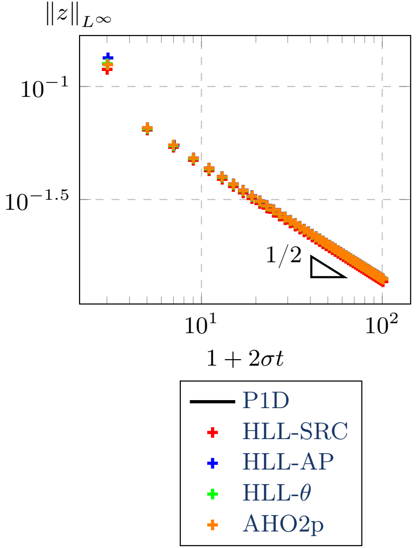




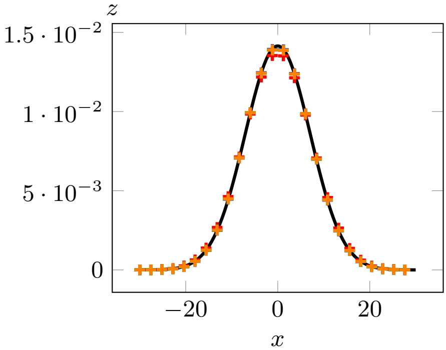
- $x_{\max} = -x_{\min} = 4$,
- $a = 1$,
- Neumann boundary conditions,
- $\sigma = 1000$, $T_f = 50$, $\Delta x = 4 \times 10^{-2}$

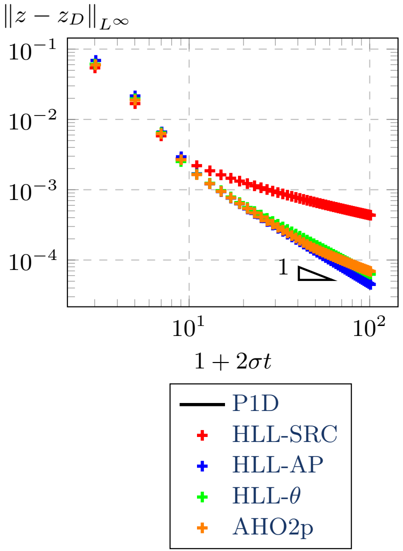
Late-time behaviour





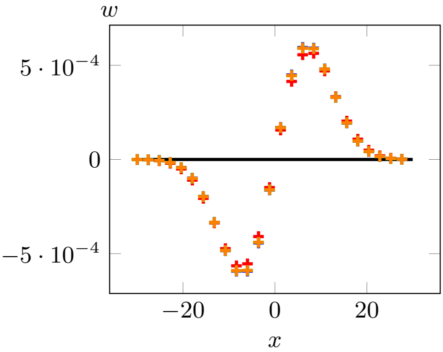
- $x_{\max} = -x_{\min} = 30$,
- $a = 1$,
- Neumann boundary conditions,
- $\sigma = 1$, $T_f = 50$, $\Delta x = 6 \times 10^{-2}$

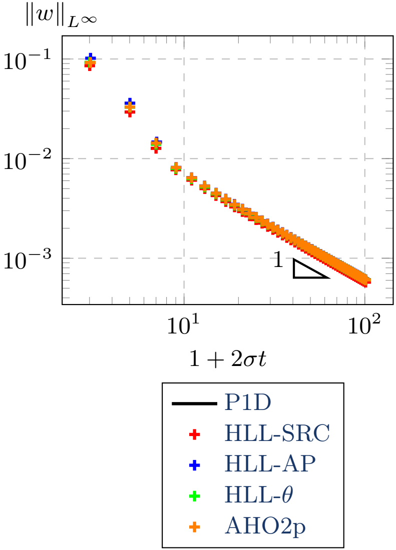




- $x_{\max} = -x_{\min} = 30$,
- $a = 1$,
- Neumann boundary conditions,
- $\sigma = 1$, $T_f = 50$, $\Delta x = 6 \times 10^{-2}$

Late-time behaviour





- $x_{\max} = -x_{\min} = 30$,
- $a = 1$,
- Neumann boundary conditions,
- $\sigma = 1$, $T_f = 50$, $\Delta x = 6 \times 10^{-2}$

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Conclusion

- which scheme is the best?

Conclusion

- which scheme is the best?
- even on a 1D linear system: many questions still opened!

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Perspectives/Questions

- with ε?

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Perspectives/Questions

- with ε?
- more schemes? more tests?

Conclusion

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Perspectives/Questions

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- more schemes? more tests?
- non-linear systems (e.g. Euler with high-friction): which schemes are still valid?

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Perspectives/Questions

- with ε?
- more schemes? more tests?
- non-linear systems (e.g. Euler with high-friction): which schemes are still valid?
- 2D unstructured meshes [BDF12; BT16]?

Conclusion and perspectives

Conclusion

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Perspectives/Questions

- with ε?
- more schemes? more tests?
- non-linear systems (e.g. Euler with high-friction): which schemes are still valid?
- 2D unstructured meshes [BDF12; BT16]?
- name of the code?

Thanks for your attention.

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